ELEC 311

History of Electromagnetics

1747 Benjamin Franklin introduces the concept of electric charge

1764 Joseph Lagrange discovers divergence thrm. Guass re-discovers in 1813 (Guass's Law)

1820 Oersted discovers relationship between current and magnetism

1825 Ampere publishes complete description of electromagnetism

1831 Faraday discovers relationship between changing magnetic flux and voltage

1837 Faraday introduces dielectric constant, in 1846 he speculates light=wave

1849 Fizeau conducts hilltop experiment and predicts speed of light (Very close)

1850 William Thomson introduces $\vec{B}$, $\vec{H}$, permeability and susceptibility

1864 Maxwell presents complete mathematical description of electromagnetism

1888 Hertz detects and generates RF waves

1900 Wireless technology achieves signifigance

1920 Broadcast radio is a consumer force

Electromagnetic Quantities

E - electric field strength (V/m)

H - magnetic field strength (A/m)

D - electric flux density (C/m^2)

B - magnetic flux density ($Wb/m^2 = T$)

J - electric current density (A/m^2)

ρ_v - electric charge density (C/m^3)

Permittivity, Permeability, Conductivity:

ε_0 - permittivity of free space (F/m)

ε_r - relative permittivity (unitless)

μ_0 - permeability of free space (H/m)

μ_r - relative permeability (unitless)

σ - Conductivity (℧/m)

Constitutive Relations (Linear Isotropic):

1) $\vec{D} = \varepsilon_r \varepsilon_0 \vec{E}$

2) $\vec{B} = \mu_r \mu_0 \vec{H}$

3) $\vec{J} = \sigma \vec{E}$

Scalar electric potential:

$$E = -\nabla V$$

$$V = \int_v \frac{\rho}{4\pi\varepsilon_0 R}\, dv$$

Vector magnetic Potential:

$$B = \nabla \times A$$

requirement: $\nabla \cdot B = 0$

$$A = \oint \frac{\mu I}{4\pi R}\, dL$$

Maxwell's unique contribution:

"Much of maxwell's theory had been previously speculated without the math to support it. His unique contribution was displacement current"

Interpreting Maxwell's equations:

1) Free electric charges exist, Free magnetic charges do not (Result of divergence eqn's)
2) Constant current gives rise to a perpendicular magnetic field
3) Time varying electric fields give rise to time varying magnetic fields and vice versa
4) The field vectors are perpendicular to each other
5) Time varying E/H fields don't exist in isolation

Maxwell's Equations

Point form:

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{dB}{dt}$$

$$\nabla \times H = J_c + \frac{dD}{dt}$$

Integral form:

$$\oint_s D \cdot dS = \int_v \rho\, dv$$

$$\oint_s B \cdot dS = 0$$

$$\oint_c H \cdot dL = \int_s \left(J_c + \frac{dD}{dt}\right) \cdot dS$$

$$\oint_c E \cdot dL = \int_s -\frac{dB}{dt} \cdot dS$$

Free Space:

$$\nabla \cdot D = 0$$

$$\nabla \cdot B = 0$$

$$\nabla \times H = \frac{dD}{dt}$$

$$\nabla \times E = -\frac{dB}{dt}$$

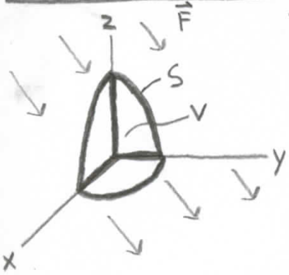
$$\oint_s D \cdot dS = 0$$

$$\oint_s B \cdot dS = 0$$

$$\oint_c E \cdot dL = \int_s \frac{-dB}{dt} \cdot dS$$

$$\oint_c H \cdot dL = \int_s \frac{dD}{dt} \cdot dS$$

The divergence thereom:



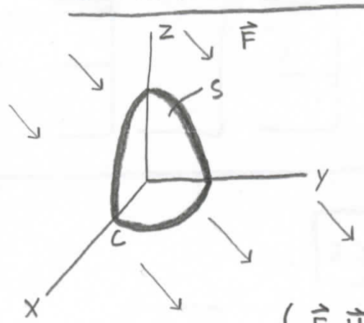
"Consider a closed surface S which encloses a volume V and you have a vector field $\vec{F}$"

$$\oint_S \vec{F}\cdot d\vec{S} = \int_V (\nabla\cdot F)\,dv$$

example: (Flux density)

$$\oint_S \vec{D}\cdot d\vec{s} = \int_V (\nabla\cdot D)\,dv = Q_{enc}$$

Stokes thereom:



"Consider an open surface S whose boundery is given by a closed curve C and you have a vector field $\vec{F}$"

$$\oint_C \vec{F}\cdot d\vec{L} = \int_S (\nabla\times\vec{F})\,d\vec{S}$$

example: (Field Strength)

$$\oint_C \vec{A}\cdot d\vec{L} = \int_S \vec{B}\cdot d\vec{S} = \Phi$$

Helmholtz's thereom:

$\nabla\cdot D =$ Something
$\nabla\cdot B =$ Something
$\nabla\times E =$ Something
$\nabla\times H =$ Something

Experimental Observations

Displacement Current:

$$J_D = \frac{dD}{dt}$$

"For non-static fields"

$$\nabla\times H = J_C + J_D$$

$$J = J_C + J_D$$
$$= \sigma E + \frac{d(\varepsilon E)}{dt}$$
$$= \sigma E + j\omega\varepsilon E$$

$$\left|\frac{J_C}{J_D}\right| = \frac{\sigma}{\omega\varepsilon} \sim \text{loss tangent}$$

(Phase difference between displacement and conduction current)

Wave Propogation

Uniform plane wave: (Lossless medium)

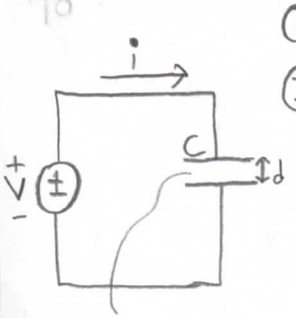
"Propogates in the $\pm z$ direction as an infinite plane"

$$E_x = Ae^{-jkz} + Be^{jkz} \text{ V/m [Horizontally polarized]}$$

$$H_y = \frac{1}{\eta_0}\left(Ae^{-jkz} + Be^{jkz}\right) \text{ A/m [Vertically polarized]}$$

1. e^{-jkz} solution → Describes a plane travelling in the $+z$ direction
2. e^{jkz} solution → Describes a plane travelling in the $-z$ direction

Displacement Current



"How does current flow through here?"

1. $V = \int \vec{E}\cdot d\vec{L} = |\vec{E}|\cdot d$ [For constant $\vec{E}$ between plates]
2. $C = \frac{\varepsilon A}{d}$ [Capacitor equation]
3. $i = C\frac{dv}{dt}$ [Current in capacitor]
4. $i = JA$ [Current density]

Rearranging:

$$J\cancel{A} = \frac{\varepsilon \cancel{A}}{\cancel{d}}\cdot\frac{d(E\cdot \cancel{d})}{dt}$$

$$\vec{J} = \frac{\varepsilon\, d\vec{E}}{dt} = \frac{d\vec{D}}{dt}$$

$$\boxed{J_D = \frac{dD}{dt}}$$

Types of Current:

① **Drift** $\vec{J} = nq\vec{v}$ "Free charges moving, ie: electron beam, particle accelerator, etc"

② **Conduction** $\vec{J} = \sigma \vec{E}$ "Common, current moving through a wire"

③ **Displacement** $\vec{J} = \frac{\partial \vec{D}}{\partial t}$ "Changing flux behaves like a current, no charges present

Math Review

$\vec{F}(x,y,z) = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ [Vector field], $f(x,y,z)$ [scalar field]

Dot product:

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = |A||B|\cos\theta$$

Cross product:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

$$|\vec{A}\times\vec{B}| = |A||B|\sin\theta$$

Gradient

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

"Points in the direction of greatest increase"

Divergence

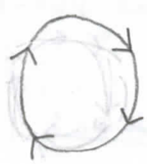
$$\nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

"Does net flux flow more in or out of a region?"

- $+ \rightarrow$ Net outflow
- $- \rightarrow$ Net inflow
- $0 \rightarrow$ What flows in flows out

Curl

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

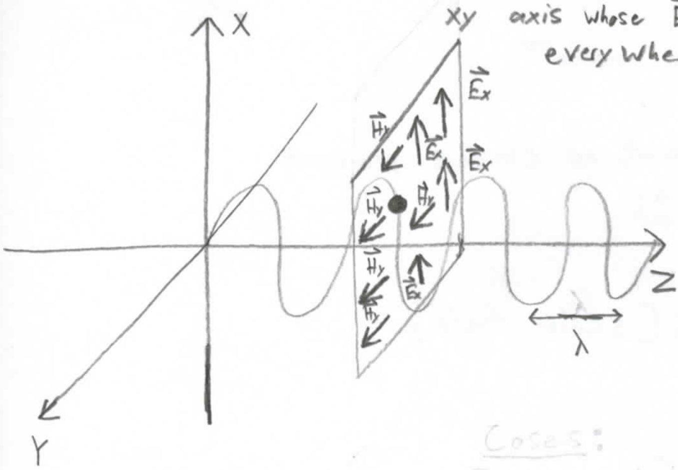


"If you dropped a stick into a region, would it rotate clockwise or counter-clockwise?" [2D]

- $+ \rightarrow$ Clockwise rotation
- $- \rightarrow$ Counter-clockwise rotation
- $0 \rightarrow$ No rotation

Plane Waves

"Wave propogates in the z-direction, at any point in space and time (z_0, t_0) there exists an infinite plane sitting in the xy axis whose $\vec{E}$ and $\vec{H}$ field are the same every where in the plane"



$$\gamma = \alpha + j\beta$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right)}$$

Cases: (Conductivity)

① Free-Space / Perfect Dielectric
② Partially conducting medium
③ Good conductor

Lossless Mediums

(free-space or perfect dielectric)

$$\alpha = 0$$

$$E_x(z,t) = E_0 e^{j(\omega t - \beta z)}$$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda} \qquad V_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}}$$

Intrinsic Impedance:

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Partially Conducting Medium

$\alpha, \beta =$ General eqn's

$$E_x(z,t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{x}$$

$$H_y(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} e^{-j(\omega t - \beta z - \theta)}\hat{y}$$

Intrinsic Impedance:

$$\eta = |\eta|\angle\theta = \frac{E_x}{H_y} \quad \text{(Now complex)}$$

Good Conductor

$$\boxed{\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma}}$$

$$E_x(z,t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)}\hat{x}$$

$$H_y(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} e^{j(\omega t - \beta z - \pi/4)}\hat{y}$$

Intrinsic Impedance:

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^\circ \quad \text{(Now complex)}$$

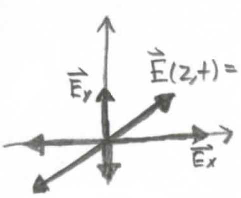
Skin depth:

$$\delta = \frac{1}{\alpha}$$

Polarization

Linear Polarization:

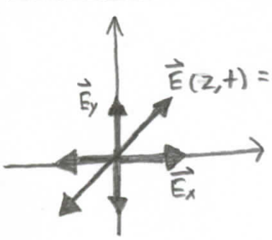
"Use superposition of components to get arbitrarily polarized $\vec{E}$ field"



$$\vec{E}(z,t) = E_x e^{j(\omega t - \beta z)}\hat{x} + E_y e^{j(\omega t - \beta z)}\hat{y}$$

Circular Polarization:

"Horizontal and vertical components are out of phase by 90°, still sum using superposition but no traces out a circle over time"



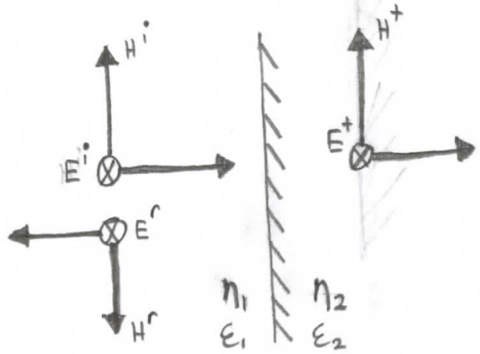
$$\vec{E}(z,t) = E_x e^{j(\omega t - \beta z)}\hat{x} + E_y e^{j(\omega t - \beta z \pm \pi/2)}\hat{y}$$

Elliptical Polarization:

$$E(z,t) = E_x e^{j(\omega t - \beta z)}\hat{x} + E_y e^{j(\omega t - \beta z \pm \theta)}$$

Wave propogation at Boundaries

Normal Incidence:



at the boundary:

(1) $E^i + E^r = E^t$

(2) $H^i - H^r = H^t$

Intrinsic Impedances:

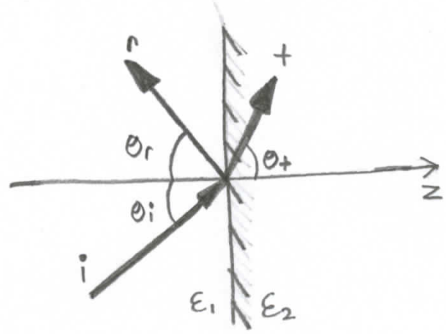
$$\eta_1 = \frac{E^i}{H^i}, \quad \eta_1 = \frac{E^r}{H^r}, \quad \eta_2 = \frac{E^t}{H^t}$$

Index of refraction:

$n = \sqrt{\varepsilon_r}$ [Not to be confused with η]

$\boxed{\text{Perpendicular or TE polarization}}$

Oblique Incidence:



Law of Reflection: $\theta_i = \theta_r$

Snell's Law: $n_i \sin\theta_i = n_t \sin\theta_t$

ELEC 311 - Electromagnetic Fields & Waves
Formula Sheet (13 Feb 2020)

$$\begin{aligned} \mathbf{D} &= \epsilon_r \epsilon_0 \mathbf{E} \\ \mathbf{B} &= \mu_r \mu_0 \mathbf{H} \\ \mathbf{J} &= \sigma \mathbf{E} \end{aligned}$$

$$\nabla \times \mathbf{A} = \mu \mathbf{H}$$

$$V = \int_v \frac{\rho\, dv}{4\pi\epsilon_0 R}$$

$$\mathbf{A} = \oint \frac{\mu I d\ell}{4\pi R}$$

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\partial \mathbf{B}/\partial t \\ \nabla \times \mathbf{H} &= \mathbf{J}_c + \partial \mathbf{D}/\partial t \end{aligned}$$

$$\mathbf{S_{avg}} = \frac{1}{2} Re\{\mathbf{E} \times \mathbf{H}^*\}$$

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_v (\nabla \cdot \mathbf{F}) dv$$

$$\oint_C \mathbf{F} \cdot d\ell = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

$$\begin{aligned} \oint_S \mathbf{D} \cdot d\mathbf{S} &= \int_v \rho\, dv \\ \oint_S \mathbf{B} \cdot d\mathbf{S} &= 0 \\ \oint_C \mathbf{H} \cdot d\ell &= \int_S \left(\mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \\ \oint_C \mathbf{E} \cdot d\ell &= \int_S -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \end{aligned}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \ \text{ F/m}$$

$$\mu_0 = 4\pi \times 10^{-7} \ \text{ H/m}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi\, \Omega$$ Impedance of free Space

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{H} &= 0 \\ \nabla \times \mathbf{E} &= -j\omega \mathbf{B} = -j\omega\mu \mathbf{H} \\ \nabla \times \mathbf{H} &= \mathbf{J}_c + j\omega \mathbf{D} = (\sigma + j\omega\epsilon)\mathbf{E} \end{aligned}$$

$$\partial^2 E_x / \partial z^2 + k^2 E_x = 0$$

$$v = \frac{\omega}{\beta} = \boxed{f\lambda = \frac{1}{\sqrt{\epsilon\mu}}}$$

$$\gamma = \alpha + j\beta$$

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right)}$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right)}$$

$$\begin{aligned} E_x &= A\, e^{-jkz} + B\, e^{+jkz} \ \text{ V/m} \\ H_y &= \frac{1}{\eta_0} \left(A\, e^{-jkz} + B\, e^{+jkz} \right) \ \text{ A/m.} \end{aligned}$$

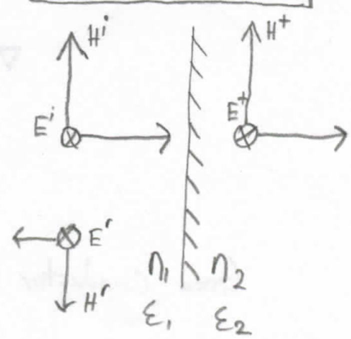
Normal Incidence

$$\frac{E_0^r}{E_0^i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$$

$$\frac{H_0^r}{H_0^i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$$

$$\frac{E_0^t}{E_0^i} = \frac{2\eta_2}{\eta_1 + \eta_2}$$

$$\frac{H_0^t}{H_0^i} = \frac{2\eta_1}{\eta_1 + \eta_2}$$



Snell's Law

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}} = \frac{v_1}{v_2}\,.$$

$$E_\theta = E_0\,\frac{e^{-jkr}}{r}$$

$$\frac{E_0^r}{E_0^i} = \Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\frac{E_0^t}{E_0^i} = \tau_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\frac{E_0^r}{E_0^i} = \Gamma_\parallel = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\frac{E_0^t}{E_0^i} = \tau_\parallel = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\theta_B = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

$$S_{\text{avg}} = \frac{1}{2}|E_\theta|^2/\eta_0$$

$$\Gamma_{TE}(E) = \Gamma_{TE}(H) = -\frac{\sin(\theta_i-\theta_t)}{\sin(\theta_i+\theta_t)}$$

$$\Gamma_{TM}(E) = \Gamma_{TM}(H) = -\frac{\tan(\theta_i-\theta_t)}{\tan(\theta_i+\theta_t)}$$

$$R_\Omega = \frac{\rho L}{A} = \frac{L}{\sigma A}$$

$$\mathbf{E} = \frac{1}{j\omega\epsilon_0}\nabla\times\mathbf{H}$$

$$\nabla\times\mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\beta = \omega\sqrt{\mu\epsilon}$$

Good Conductor $\delta = \dfrac{1}{\sqrt{\pi f\mu\sigma}}$, $\alpha = \beta = \sqrt{\pi f \mu \sigma}$

$$E_x = E_0\,e^{j(\omega t-\beta z)}$$

$$\sin t = \cos(t-\pi/2)$$

$$\sin^2 t + \cos^2 t = 1$$

$$\sin 2t = 2\sin t\cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1 = 1 - 2\sin^2 t$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$2\cos^2(a/2) = 1 - \cos a$$

$$2\sin^2(a/2) = 1 + \cos a$$

$\eta_0 = \sqrt{\dfrac{\mu_0}{\epsilon_0}} = 120\pi\Omega$ Free space

$\eta = \sqrt{\dfrac{\mu}{\epsilon}}$ Perfect Dielectric

$\eta = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}}$ Partial Conductor

$\eta = \sqrt{\dfrac{\omega\mu}{\sigma}}\angle 45^\circ$ Good Conductor

$$\eta_w(z) = \eta_1\,\frac{\eta_2 - j\eta_1\tan\beta_1 z}{\eta_1 - j\eta_2\tan\beta_1 z}$$

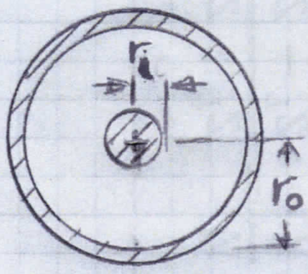
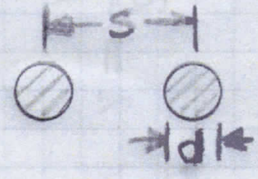
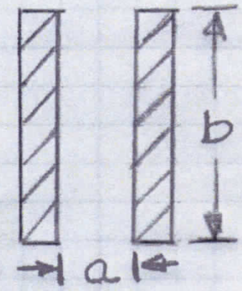
$$\tan\delta = \frac{\sigma}{\omega\epsilon}$$

$$v_g = v\sin\theta_m$$

$$v_g v_p = v^2$$

$$k^2 = k_c^2 + k_z^2$$

ELEC 311 — TRANSMISSION LINES — FORMULA SHEET

	COAXIAL LINE	LADDER LINE	PARALLEL PLATES
μ, ϵ, σ ARE PROPERTIES OF THE DIELECTRIC		IF $s \gg d$ $\cosh^{-1} x \approx \ln(2x)$ 	$a \ll b$ 
C F/m	$\dfrac{2\pi\epsilon}{\ln\left(\frac{r_o}{r_i}\right)}$	$\dfrac{\pi\epsilon}{\cosh^{-1}\left(\frac{s}{d}\right)}$	$\dfrac{\epsilon b}{a}$
L H/m	$\dfrac{\mu}{2\pi}\ln\left(\dfrac{r_o}{r_i}\right)$	$\dfrac{\mu}{\pi}\cosh^{-1}\left(\dfrac{s}{d}\right)$	$\mu\dfrac{a}{b}$
G S/m	$\dfrac{2\pi\sigma}{\ln\left(\frac{r_o}{r_i}\right)}$	$\dfrac{\pi\sigma}{\cosh^{-1}\left(\frac{s}{d}\right)}$	$\sigma\dfrac{b}{a}$
R Ω/m	$\dfrac{R_s}{2\pi}\left(\dfrac{1}{r_o}+\dfrac{1}{r_i}\right)$	$\dfrac{2R_s}{\pi d}\left[\dfrac{s/d}{\sqrt{(s/d)^2-1}}\right]$	$\dfrac{2R_s}{b}$
Z_o Ω	$\dfrac{\eta}{2\pi}\ln\left(\dfrac{r_o}{r_i}\right)$	$\dfrac{\eta}{\pi}\cosh^{-1}\left(\dfrac{s}{d}\right)$	$\eta\dfrac{a}{b}$
Z_o IN AIR	$60\ln\left(\dfrac{r_o}{r_i}\right)$	$120\cosh^{-1}\left(\dfrac{s}{d}\right)$	$120\pi\dfrac{a}{b}$
α_c Np/m	$\longleftarrow \dfrac{R}{2Z_o} \longrightarrow$ DUE TO CONDUCTOR		
α_d Np/m	$\longleftarrow \dfrac{GZ_o}{2} \longrightarrow$ DUE TO DIELECTRIC		
α dB/m	$\longleftarrow 8.686(\alpha_c+\alpha_d) \longrightarrow$ TOTAL ATTENUATION		
β rad/m	$\longleftarrow \omega\sqrt{\mu\epsilon} = 2\pi/\lambda \longrightarrow$ FOR LOW LOSS LINES		

σ_c & δ ARE PROPERTIES OF THE CONDUCTOR. $R_s = 1/\sigma_c\delta$ = SKIN EFFECT SURFACE RESISTIVITY